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A CONSTRUCTIVE MODEL OF IMPACT WITH FRICTION*

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The stereomechanical impact of rough bodies is described by a model based on specifying the impact forces as a function of the deformations (e.g., by analogy with the properties of a visco-elastic medium). The coefficients of friction and restitution relative to the velocity are not specified in advance, so that their dependence on the initial conditions can be studied. Earlier, impact with friction was studied /1-3/ by means of a formally axiomatic approach. The present model is simpler and in better agreement with experimental data (e.g., /4/).

1. Formulation of the problem. We consider a mechanical system in configuration space $\mathbf{x} \in R^n$, with kinetic energy $T = {}^1/_2\mathbf{x}^{\cdot} A(\mathbf{x})\mathbf{x}^{\cdot T}$, $A \in R^{n^2}$, generalized forces $\mathbf{Q}(\mathbf{x}, \mathbf{x}^{\cdot}) \in R^n$, and positional relation $f(\mathbf{x}) \geqslant 0$. For a system of two rigid bodies, the latter means that points of the bodies cannot simultaneously occupy the same positions in space; f is the distance between the bodies.

If, at an instant $t = t^\circ$ we have

$$f(\mathbf{x}) = 0, \quad f^{\cdot}(\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial f}{\partial \mathbf{x}_j} \mathbf{x}_j^{\cdot} < 0$$

impact occurs in the system. We make the assumptions of stereomechanical theory /5/, that we can neglect both the duration of the impact and the accompanying wave processes. Then, the pre- and post-impact values of the coordinates $\mathbf{x}_-$ and $\mathbf{x}_+$ are the same, and description of the impact amounts to finding the dependence of $\mathbf{x}_+^{\cdot}$ and $\mathbf{x}_-^{\cdot}$.

Lagrange's equations for the impact are /1/

$$\Delta\mathbf{x}^{\cdot} A(\mathbf{x}^\circ) = \mathbf{I}, \quad \mathbf{x}^\circ = \mathbf{x}(t^\circ), \quad \Delta\mathbf{x}^{\cdot} = \mathbf{x}_+^{\cdot} - \mathbf{x}_-^{\cdot} \tag{1.1}$$

where $\mathbf{I}$ is the impact momentum.

The determination of $\mathbf{I}$ is usually based on certain postulates /1-3/. They include Newton's hypothesis that the impact has two phases, in which

$$(\mathbf{x}_+^{\cdot}, \mathbf{N}) = -\varkappa(\mathbf{x}_-^{\cdot}, \mathbf{N}), \quad 0 \leqslant \varkappa \leqslant 1 \tag{1.2}$$

where $\varkappa$ is the coefficient of restituion, and $\mathbf{N}$ is the normal vector to the surface $f(\mathbf{x}) = 0$ at the point $\mathbf{x}^\circ$. The second postulate, about the Coulomb nature of the impact friction, asserts that, during the entire impact,the normal and tangential stresses are connected by Coulomb's law /1/. Together, these two assumptions enable the dependence of $\mathbf{x}_+^{\cdot}$ and $\mathbf{x}_-^{\cdot}$ to be uniquely defined /1-3/, though the dependence is so complicated that explicit expressions can only be obtained in the simplest special cases /6/.

Another approach to describing stereomechanical impact is to use physical models of the impact forces. The condition $f(\mathbf{x}) \geqslant 0$ is then assumed to be violated during the impact (so that $-f(\mathbf{x}) \sim \varepsilon$, $\varepsilon \ll 1$), which corresponds to defromation of the bodies. For $f < 0$ the impact forces are defined as functions of the deformations. Termination of the impact corresponds to a change of sign of the function f from minus to plus. This approach is useful when the axiomatic description is ill-posed /7/.

A constructive approach was used in /8/ to describe the impact of smooth bodies. Our present task is to study impact with friction on this basis.

2. Description of the model. Since the special features of the body shape and motion may determine the choice of generalized coordinates, the latter may in general be non-orthogonal /9/. Thus, the metric concepts of orthogonality and norm are defined in the sense of the scalar product

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}A(\mathbf{x}^\circ)\mathbf{b}^T, \quad \mathbf{a}, \mathbf{b} \in R^n \tag{2.1}$$

which is invariant under the choice of the generalized coordinates.

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The normal vector to the surface $f(\mathbf{x}) = 0$ at the point $\mathbf{x}^\circ$ is $\mathbf{N} = \operatorname{grad} f \cdot A^{-1}(\mathbf{x}^\circ)$, while the tangential and normal components of the impact momentum are given by

$$\mathbf{I}_n = (\mathbf{I}, \mathbf{n})\,\mathbf{n}, \quad \mathbf{I}_t = \mathbf{I} - \mathbf{I}_n, \quad \mathbf{n} = \mathbf{N}/|\mathbf{N}|, \quad |\mathbf{N}| = (\mathbf{N}, \mathbf{N})^{1/2}$$

The surfaces of rough bodies are usually divided into macro- and microstructures /10/, where the former depends on the geometrical shape, while the latter is only known statistically. We can then write the positional relation f in the form

$$f = \frac{f_*^\circ - g}{|\mathbf{N}^*|}, \quad f_*^\circ = \frac{f^\circ}{|\mathbf{N}^\circ|}, \quad \mathbf{N}^\circ = \operatorname{grad} f^\circ \cdot A^{-1}(\mathbf{x}^\circ), \qquad (2.2)$$
$$\mathbf{N}^* = \operatorname{grad}(f_*^\circ - g)\, A^{-1}(\mathbf{x}^\circ)$$

where the basic surface is given by the equation $f^\circ = 0$ (or in normalized form, $f_*^\circ = 0$), while the microrelief is given by the function $g(\mathbf{x})$, which is randomly chosen from a functional space G with probability measure P; we shall assume that $|g(\mathbf{x})| \leqslant \varepsilon_1 \ll 1$, $g \in G$. The meaning of the normalization in (2.2) will be explained later, when obtaining Eq.(2.4).

Let us compare the mutual arrangement of the surfaces $f = 0$ and $f^\circ = 0$. If $f^\circ(\mathbf{x}^\circ) = 0$, then by Taylor's formula

$$f^\circ(\mathbf{x}^\circ + \mathbf{h}) = \sum_{j=1}^{n} h_j \frac{\partial f^\circ}{\partial x_j}\Big|_{\mathbf{x}^\circ} + o(|\mathbf{h}|) = (\mathbf{h}, \mathbf{N}^\circ) + o(|\mathbf{h}|)$$

The equation $f(\mathbf{x}^\circ + \mathbf{h}) = 0$, i.e., $f_*^\circ(\mathbf{x}^\circ + \mathbf{h}) = g(\mathbf{x}^\circ + \mathbf{h})$, takes the form

$$(\mathbf{h}, \mathbf{N}^\circ/|\mathbf{N}^\circ|) = g(\mathbf{x}^\circ + \mathbf{h}) + o(|\mathbf{h}|)$$

Thus, g describes the height of the rough surface profile above the basic profile, calculated in the direction of the normal to the latter.

To find the generalized forces with $f < 0$ we use the Kelvin-Voigt /5/ model to a visco-elastic medium, putting

$$\mathbf{Q}^* = -(M^2 f + 2kMf^{\cdot}) \operatorname{grad} f, \quad 0 \leqslant k \leqslant 1, \quad M \gg 1 \qquad (2.3)$$

Neglecting in the equations of motion with $f < 0$ the finite generalized forces $\mathbf{Q}$ and the small orders $\varepsilon, \varepsilon_1$, and using the normalization of (2.2), we obtain the relation

$$f^{\cdot\cdot} = (\mathbf{x}^{\cdot\cdot}, \mathbf{N}) = (\mathbf{Q}^* A^{-1}(\mathbf{x}^\circ), \mathbf{N}) = -(M^2 f + 2kMf^{\cdot}) \qquad (2.4)$$

On solving (2.4) under the initial conditions $f(t^\circ) = 0$, $^{\cdot}(t^\circ) = f_\circ^{\cdot}$, we have

$$f = f_\circ^{\cdot} M^{-1}(1 - k^2)^{-1/2} \sin[M(1 - k^2)^{1/2}(t - t^\circ)] \exp[-kM(t - t^\circ)] \qquad (2.5)$$

Corresponding to the interval of impact $t^\circ \leqslant t \leqslant t^\circ + \tau$, we have $f < 0$, so that $\tau = \pi M^{-1}(1 - k^2)^{-1/2}$.

We find $\mathbf{I}$ in (1.1) as the mean value

$$\mathbf{I} = \int_{t^\circ}^{t^\circ + \tau} dt \int_G \mathbf{Q}^*(g, t)\, dP(g, \mathbf{x}_-^{\cdot}) \qquad (2.6)$$

Note that $\mathbf{Q}^*(g, t)$ is given by (2.3), (2.5), while $dP(g, \mathbf{x}_-^{\cdot})$ is the probability, which depends on $\mathbf{x}_-^{\cdot}$ of the trajectory $\mathbf{x}(t)$ intersecting the surface $f(\mathbf{x}) = 0$ at the point $\mathbf{x}^\circ$ for a given realization of the microrelief $g \in G$. The nature of this dependence can be seen from the following arguments. First, the initial conditions of impact $\mathbf{x}(t^\circ) = \mathbf{x}^\circ$, $\mathbf{x}^{\cdot}(t^\circ) = \mathbf{x}_-^{\cdot}$ are in accord, not with all the realizations $g \in G$, but only with those for which $(\mathbf{N}, \mathbf{x}_-^{\cdot}) < 0$ (since $f > 0$ before impact). Second, depending on the angle between the vectors $\mathbf{x}_-^{\cdot}$ and $\mathbf{N}$, the projection ds^* of the elementary area ds of the tangent plane to the surface $f = 0$ at the point $\mathbf{x}^\circ$ varies in the direction $\mathbf{x}_-^{\cdot}$: if $\mathbf{x}_-^{\cdot}$ and $\mathbf{N}$ are collinear, this projection is a maximum, while as $\mathbf{N}$ varies it decreases in accordance with the relation $ds^* = -(\mathbf{N}, \mathbf{x}_-^{\cdot})\,|\mathbf{x}_-^{\cdot}|^{-1} ds$.

On normalizing $P(g, \mathbf{x}_-^{\cdot})$ in such a way that it becomes a probability measure in the set G^* of admissible realizations of the microrelief, we obtain the final expression for this function:

$$dP(g, \mathbf{x}_-^{\cdot}) = (\mathbf{N}, \mathbf{x}_-^{\cdot})\, dP(g) \Big/ \int_{G^*} (\mathbf{N}, \mathbf{x}_-^{\cdot})\, dP(g) \qquad (2.7)$$
$$G^* = G \cap \{g \mid (\mathbf{N}, \mathbf{x}_-^{\cdot}) < 0\}$$

On subsituting (2.3), (2.5) and (2.7) into (2.6) and noting that $f_\circ^{\cdot} = (\mathbf{N}, \mathbf{x}_-^{\cdot})$, we arrive at the following equation of impact with friction:

$$\Delta\mathbf{x}^{\cdot} = -(1 + e) \int_{G^*} (\mathbf{N}, \mathbf{x}_-^{\cdot})^2\, \mathbf{N}\, dP(g) \Big/ \int_{G^*} (\mathbf{N}, \mathbf{x}_-^{\cdot})\, dP(g) \qquad (2.8)$$

$$e = \exp\left[-k\pi(1-k^2)^{-1/2}\right], \quad 0 \leqslant e \leqslant 1$$

To analyse (2.8), some assumptions are needed about the microrelief, i.e., about the properties of the probability space (G, P). We assume that it is isotropic, i.e., invariant under rotations relative to the vector n° of the normal to the surface $f^\circ(\mathbf{x}) = 0$.

This property implies identities that enable (2.8) to be simplified. We write $\mathbf{N}$ as the sum $N = \mathbf{n}^\circ \cos\alpha + \mathbf{l}\sin\alpha$, where $(\mathbf{l}, \mathbf{n}^\circ) = 0$, $|\mathbf{l}| = 1$, $\alpha = \widehat{\mathbf{N}\,\mathbf{n}^\circ}$. We then have

$$\int_G \mathbf{l}(g) F[\alpha(g)]\, dP(g) = 0, \quad F \in C^0(R) \tag{2.9}$$

$$\int_G (\mathbf{y}, \mathbf{l}(g))^2 F[\alpha(g)]\, \mathbf{l}(g)\, dP(g) = 0, \quad \mathbf{y} \in R^n$$

To prove this, we need only note that, by the isotropic property, if, corresponding to the functions $g_1, g_2 \in G$, we have the curves $f_{1,2} = 0$, obtained by a mutual rotation of 180° relative to the vector $\mathbf{n}^\circ$ (here, $\mathbf{l}(g_1) = -\mathbf{l}(g_2)$, $\alpha(g_1) = \alpha(g_2)$), then $dP(g_1) = dP(g_2)$.

If the angle of attack $\beta = \operatorname{arctg}(|\mathbf{v}_l| / |\mathbf{v}_n|)$ is sufficiently small, so that $\max_G |\alpha(g)| < \pi/2 - \beta$, and $G^* = G$, then, using (2.9), we obtain Eq.(2.8) in the form

$$\Delta\mathbf{x}^{\cdot} = -\frac{1+e}{\int_G \cos\alpha\, dP(g)} \left\{\mathbf{n}^\circ \left[\frac{\mathbf{v}_l^2}{|\mathbf{v}_n|} \int_G \left(\frac{\mathbf{v}_l}{|\mathbf{v}_l|}, \mathbf{l}\right)^2 \sin^2\alpha\cos\alpha\, dP(g) + \right.\right. \tag{2.10}$$

$$\left.\left. |\mathbf{v}_n| \int_G \cos^3\alpha\, dP(g)\right] + 2\int_G (\mathbf{v}_l, \mathbf{l}) \sin^2\alpha\cos\alpha\, \mathbf{l}\, dP(g)\right\}$$

$$\mathbf{v} = \mathbf{x}_-^{\cdot}, \quad \mathbf{v}_n = (\mathbf{v}, \mathbf{n}^\circ)\mathbf{n}^\circ, \quad \mathbf{v}_l = \mathbf{v} - \mathbf{v}_n$$

where $\mathbf{v}_n, \mathbf{v}_l$ are the normal and tangential components of the initial velocity with respect to the basic surface $f^\circ = 0$.

For large angles β, such that $G^* \neq G$, supplementary terms appear in Eq.(2.10), so that in this case it is better to use (2.8) directly.

Example. Consider the case of plane impact $(n = 2)$. We choose the coordinates so that the basic surface has the equation $x_2 = 0$. For simplicity, we shall assume that the set G consists of piecewise linear functions $x_2 = g(x_1)$, while at points of differentiability, $|g'(x_1)| = \operatorname{tg}\alpha = \text{const}$. The isotropic property here means that on average the number of "lifts" and "drops" of the curve $x_2 = g(x_1)$ are the same.

If the impact is along the normal to the basic surface, $\mathbf{v}_l = 0$, then only the second term in (2.10) is non-zero, so that we have

$$\Delta\mathbf{x}^{\cdot} = -(1+e)\,|\mathbf{v}_n|\cos^2\alpha\cdot\mathbf{n}^\circ \tag{2.11}$$

For oblique impact, $\mathbf{v}_l \neq 0$. If we have $\beta < \pi/2 - \alpha$, then $G^* = G$ and $\Delta\mathbf{x}^{\cdot}$ is given by (2.10):

$$\Delta\mathbf{x}^{\cdot} = -(1+e)\left[\mathbf{n}^\circ(v_l^2|\mathbf{v}_n|^{-1}\sin^2\alpha + |\mathbf{v}_n|\cos^2\alpha) + 2\mathbf{v}_l\sin^2\alpha\right] \tag{2.12}$$

If $\beta < \pi/2 - \alpha$, then the trajectory $\mathbf{x}(t)$ can only hit the surface $f(\mathbf{x}) = 0$ in the case of those functions $g \in G$ for which $g'(\mathbf{x}^\circ) = -\operatorname{tg}\alpha\operatorname{sgn}\mathbf{v}_l$. This impact is similar to impact on a smooth inclined plane and is described by

$$\Delta\mathbf{x}^{\cdot} = -(1+e)(|\mathbf{v}_l|\cos\alpha - |\mathbf{v}_n|\sin\alpha)(-\sin\alpha\operatorname{sgn}\mathbf{v}_l, \ \cos\alpha) \tag{2.13}$$

3. Properties of the model. Let us state the main properties of impact with friction, resulting from (2.8).

1°. The direction of the impact forces remains unchanged during impact: it depends on the direction of the vector $\mathbf{v}$ but not on its modulus. A similar property is inherent in the axiomatic model /1-3/, though in the context of this the impact forces can change direction during impact. The contructive model is therefore simpler.

2°. For isotropic surfaces, in the case $\mathbf{v}_l = 0$ we have $\Delta\mathbf{x}_l^{\cdot} = 0$ from (2.10). If $\mathbf{v}_l \neq 0$, the vectors $\mathbf{v}_l$ and $\Delta\mathbf{x}_l^{\cdot}$ are collinear, in opposite directions.

For the proof of this, we will show that the vector $\Delta\mathbf{x}^{\cdot}$, given by (2.8), lies in the plane $\Pi(\mathbf{v})$ through the vectors $\mathbf{v}$ and $\mathbf{n}^\circ$. In fact, the set G can be divided into pairs of functions $g_{1,2}$ such that the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are symmetric about the plane $\Pi(\mathbf{v})$, while the surfaces $f_1 = 0$ and $f_2 = 0$ can be obtained from one another by a rotation about the vector $\mathbf{n}^\circ$; by the isotropic property, $dP(g_1) = dP(g_2)$. Since $(\mathbf{v}, \mathbf{n}_1) = (\mathbf{v}, \mathbf{n}_2)$, the two functions $g_{1,2}$ belong simultaneously to G^*, while $\mathbf{n}_1 + \mathbf{n}_2 \in \Pi(\mathbf{v})$, whence $\Delta\mathbf{x}^{\cdot} \in \Pi(\mathbf{v})$. We see that $\mathbf{v}_l$ and $\Delta\mathbf{x}_l^{\cdot}$ are in opposition because $(\mathbf{v}, \mathbf{N})^2$ is greater in the case when $\mathbf{v}_l$ and $\mathbf{l}$ are in opposition than for the same surface rotated through 180° (see also (2.12) and (2.13)).

3°. In the case of direct impact $(\mathbf{v}_l = 0)$, we have Newton's hypothesis: the coefficient of restitution of the relative velocity $\varkappa$ in (1.2) is independent of $|\mathbf{v}_n|$, while, by (2.10), $\varkappa$ depends, not only on the parameter e, which describes the viscoelastic properties of the deformations, but also on the nature of the microrelief (thus the quantity $\varkappa$ in (2.11) is equal to e when $\alpha \equiv 0$, and falls as α increases).

Notice that the independence of $\varkappa$ on v_n is due to our choice of model of the impact forces in (2.3); for different models this property may not hold.

4°. With $v_l \neq 0$, the coefficient of restitution $\varkappa$ depends on the angle of attach β. It follows from (2.10) that, with $|v_n|$ constant, $\varkappa$ increases as $|v_l|$ increases; this property of the model is in accord with experimental data /11/. Thus the constructive model is more realistic than the axiomatic model in this sense (the latter is based on the fact that $\varkappa$ is independent of the initial data).

Notice that, for sufficiently large values of the angle β the coefficient $\varkappa$ can, by (2.8), even be greater than unity. This effect is confirmed in practice by the appearance of normal displacements due to the micro-impacts that accompany the relative sliding of rough bodies, see /12/.

5°. Impact friction is due to the deviation of the vector $\mathbf{N}$ from $\mathbf{n}^\circ$: if there were no micro-relief, than $g \equiv 0$ and (2.8) would take the form

$$\Delta \mathbf{x}^{\cdot} = -(1+e)(\mathbf{v}, \mathbf{n}^\circ)\,\mathbf{n}^\circ$$

which is the same as the equation of impact of smooth bodies /1/.

To find the coefficient of impact friction μ we have to multiply the $\Delta \mathbf{x}^{\cdot}$ given by (2.8) by the matrix $A(\mathbf{x}^\circ)$ in order to obtain $\mathbf{I}$ in accordance with (1.1); we then obtain $\mu = |\mathbf{I}_l|/|\mathbf{I}_n|$. In the general case, therefore, μ depends on the properties of the matrix $A(\mathbf{x}^\circ)$, so that it is difficult to study the dependence of μ on the initial conditions of impact. In the present paper we confine ourselves to the elementary case of the impact of a particle with a rough plane, when $A(\mathbf{x}^\circ)$ is proportional to the identity matrix. Then, μ is given by

$$\mu = |\Delta v_l| \,/\, |\Delta v_n| \tag{3.1}$$

As applied to the example of Sect.2, Eq.(3.1) leads to these values of the coefficient of friction:

$$\mu = \begin{cases} 2\,\mathrm{tg}^2\alpha\,\mathrm{tg}\,\beta/(\mathrm{tg}^2\alpha\,\mathrm{tg}^2\beta+1), & \beta < \pi/2-\alpha \\ \mathrm{tg}\,\alpha, & \beta \geqslant \pi/2-\alpha \end{cases}$$

In Fig.1 we plot the curve $\mu = \mu(\mathrm{tg}\beta)$ (the continuous curve).

The drop in the coefficient of friction as β increases, when the set consists of continuously *differentiable functions, can be explained by the "shading" of the pieces of profile* with the greatest slope (i.e., by their inaccessibility for trajectories with an angle of incidence close to $\pi/2$), see Fig.2, and the broken curve of Fig.1.

4. Stability of a particle on a vibrating surface.

As an application of the constructive model of impact with friction, we consider the problem of the stability of motion of a particle that runs periodically along the axis of symmetry of a rough surface of revolution, that performs harmonic oscillations along the vertical /6/. For our model of the friction, the existence of this periodic motion follows from Property 2 of Sect.3: as the particle moves along the axis, the impact is direct, so that $\Delta \mathbf{x}_l^{\cdot} = 0$.

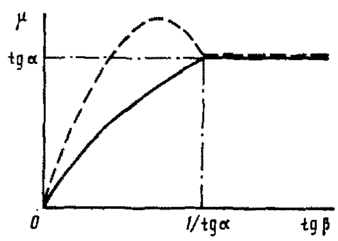

Fig.1

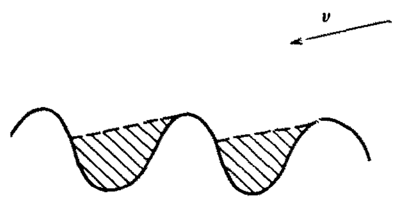

Fig.2

Assume that the disturbances of the initial conditions lead to a small deviation of the particle from the vertical. We use Eqs.(2.10) to describe the impact. By these equations, the variation of the tangential component of the velocity is given by

$$\Delta \mathbf{x}_l^{\cdot} = -\lambda \mathbf{v}_l \tag{4.1}$$

$$\lambda = 2(1+e)\left|\int_G (\mathbf{a}_l, \mathbf{l})\,\mathbf{l}\sin^2\alpha\cos\alpha\, dP(g)\right| \Big/ \int_G \cos\alpha\, dP(g)$$

where $\mathbf{a}_l$ is a unit vector tangential to the basic surface.

The form of (4.1) is the same as when describing impact in the presence of viscous friction /6/, except that here, λ may be greater than unity.

If we discard the first term in (2.10), which is of second order with respect to the disturbances, we obtain for the coefficient of restitution $\varkappa$:

$$1+\varkappa=(1+e)\int_G \cos^3\alpha\, dP(g) \Big/ \int_G \cos\alpha\, dP(g) \qquad (4.2)$$

From (4.2), the impacts are of an elastic type only when the right-hand side is greater than or equal to unity. It can be shown that, in this case, λ in (4.1) is not greater than two.

The stability conditions for the periodic motions of a particle in the presence of friction of the type (4.1) are obtained in /6/. They amount to taking a phase of the surface oscillations in which collisions occur, and to the inequality

$$0<\rho^{-1}<B, \quad B=\frac{(2-\lambda)(1+\varkappa)}{2\pi^2 m^2(1+\varkappa-\lambda)} \qquad (4.3)$$

where m is the ratio of the period of the periodic motion to the period of the surface oscillation, and ρ is the radius of curvature at a point lying on the axis of symmetry. As a result of (4.3), if $\lambda<1+\varkappa$, then B increases with λ and the class of stable cases is accordingly widened. If $1+\varkappa<\lambda\leqslant 2$, then $B\leqslant 0$ and the periodic motions are unstable whatever the shape of the support surface.

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